1 Area of a Parallelogram

Suppose we want to paint a parallelogram. How much paint do we need? Let's see how to express the answer in terms of Clifford algebra (aka geometric algebra).

For an introduction to Clifford algebra, and its application to geometry and physics, see reference 1, reference 2, reference 3, and reference 4.

The edges of the parallelogram are the vectors A and B. We're pretty sure

- the amount of paint is a scalar,
- it is a symmetric function of A and B, and
- it is positive.

In geometric algebra, we are told that the wedge product $A \wedge B$ represents area. But it's not directly the answer to the paint problem. In particular

- $A \wedge B$ is not a scalar; it's some higher-dimensional geometric abstraction.
- it is antisymmetric: $A \wedge B = -B \wedge A$
- we can't even define a ">" operator for bivectors, so we certainly can't say $A \wedge B$ will be ">" zero.

The answer is that we should be asking about the *norm* of $A \wedge B$, written $||A \wedge B||$.

We shall show that

$$\text{area} = ||A \wedge B|| = ||A||||B||| \sin(\theta)| \tag{1}$$

where θ is the angle between the vectors. This is the correct answer to the paint problem. Let's see where this result comes from.

For any multivector M, the square of the norm is given by

$$||M||^2 := MM \sim \tag{2}$$

where $M \sim$ is the reverse of M, formed by writing the vectors that make up M in reverse order. In particular

$$(A \wedge B) \sim= (B \wedge A) \tag{3}$$

And for a simple (grade=1) vector

$$\begin{array}{l} A \sim= A \\ AA \sim= AA = A.A = ||A||^2 \end{array} \tag{4}$$

Assuming we are dealing with spacelike vectors, we can write

$$A = ||A||a' \tag{5}$$

where a' is a unit vector in the direction of A.

We can write some other vector B as

$$B = ||B||(a' \cos(\theta) + a" \sin(\theta)) \tag{6}$$

where $a"$ is some unit vector perpendicular to A. Then just take these expressions for A and B, plug them into the definition of $||A \wedge B||$ and turn the crank. We find

$$\begin{array}{rcl} ||A \wedge B||^2 & = & (A \wedge B)(B \wedge A) \\ & = & ||A||^2||B||^2 \sin^2(\theta)(a' \wedge a")(a" \wedge a') \end{array} \tag{7}$$

where the cosine terms have vanished because $a' \wedge a' = 0$.

We next use the fact that the wedge product $(a' \wedge a")$ is equal to the geometric product $(a'a")$ since the two factors are orthogonal; see equation 17. This is helpful because the geometric product of geometric products is simpler than the geometric product of wedge products. In this case $(a' \wedge a")(a" \wedge a') = a'a"a"a' = 1$ and the result follows immediately:

$$\text{area} = ||A \wedge B|| = ||A||||B||| \sin(\theta)| \tag{8}$$

A more general way of taking the geometric product of two wedge products is to expand both wedge products in terms of geometric products according to equation 16, and then multiplying everything term-by-term. In this case it comes to the same thing, because

$$\begin{array}{rcl} (a' \wedge a")(a" \wedge a') & = & (a'a" - a"a')(a"a' - a'a")/4 \\ & = & (a'a"a"a' - a'a"a'a" - a"a'a"a' + a"a'a'a")/4 \\ & = & (1 + 1 + 1 + 1)/4 \end{array} \tag{9}$$

where we have used the anticommutation relation equation 14. (Sooner or later we have to exploit the fact that a' and $a"$ are perpendicular.)

This is a good homework exercise for building confidence in the formalism.

2 Volume of a Parallelepiped

Let's proceed to an even more interesting problem.

Suppose we have a parallelepiped with edges A, B and C, and we want to calculate the volume.

The old-fashioned way to do this would be to use the "triple scalar product" $A \cdot B \times C$. But cross products are bad news, and we would be much better off using the geometric algebra formulation. The desired expression is

$$\text{volume} = ||A \wedge B \wedge C|| \tag{10}$$

Note that even though we have been using the wedge product to get rid of cross products, it is not a one-for-one replacement. You cannot blindly replace $A \cdot B \times C$ by $A \cdot B \wedge C$. The triple scalar product is properly written $||A \wedge B \wedge C||$ with two wedge products and no dot products. This has a nice geometric interpretation: $A \wedge B$ is visualized as sweeping A in the direction of B, using it as a brush to sweep out the area $A \wedge B$. Similarly $A \wedge B \wedge C$ is visualized as sweeping the area $A \wedge B$ in the direction of C, using it to sweep out the volume $A \wedge B \wedge C$.

We can verify that the magnitude of the volume behaves as advertised, using the same procedure as before:

$$\begin{array}{rcl} A & = & ||A||a' \\ B & = & ||B||(a' \cos(\theta) + a" \sin(\theta)) \\ C & = & ||C||(ab' \cos(\phi) + ab" \sin(\phi)) \end{array} \tag{11}$$

where ab' is any unit vector in the AB plane, and $ab"$ is a unit vector perpendicular to the AB plane. The result is:

$$||A \wedge B \wedge C|| = ||A||||B||||C||| \sin(\theta)|| \sin(\phi)| \tag{12}$$

which makes sense.

3 Remarks

We have obtained these results without establishing any basis vectors and without expanding A, B, and C in terms of components. (That's a sign that we're doing something right.) All you need are the basic axiomatic properties

$$AB = BA \qquad \text{if A and B are colinear} \tag{13}$$

$$AB = -BA \qquad \text{if A and B are perpendicular} \tag{14}$$

$$A \cdot B = (AB + BA)/2 \tag{15}$$

$$A \wedge B = (AB - BA)/2 \tag{16}$$

$$AB = A \cdot B + A \wedge B \tag{17}$$

everywhere assuming A and B are plain old grade=1 vectors.

By the way, even the slightly-arbitrary expansion in terms of sine and cosine can be dispensed with, if we have an axiomatic definition of what "rotation" means, but that's a more-advanced topic. See reference 5.

4 References

1. Stephen Gull, Anthony Lasenby, and Chris Doran, "The Geometric Algebra of Spacetime"
   http://www.mrao.cam.ac.uk/~clifford/introduction/intro/intro.html

2. Richard E. Harke, "An Introduction to the Mathematics of the Space-Time Algebra"
   http://www.harke.org/ps/intro.ps.gz

3. David Hestenes, "Oersted Medal Lecture 2002: Reforming the Mathematical Language of Physics"
   Abstract: http://geocalc.clas.asu.edu/html/Overview.html Full paper:
   http://geocalc.clas.asu.edu/pdf/OerstedMedalLecture.pdf

4. John Denker, "Introduction to Clifford Algebra" www.av8n.com/physics/clifford-intro.htm

5. John Denker, "Multi-Dimensional Rotations, Including Boosts"
   www.av8n.com/physics/rotations.htm
   and in particular the section on "calculations."