1 Comparing Complex Numbers to Clifford Algebra

As the saying goes, learning proceeds from the known to the unknown.

There are at least two possiblities:

1) If you are starting from scratch, a nice logical path would be to learn things in the following order:
- Plain old numbers (scalars).
- Vectors.
- Clifford Algebra in two dimensions.
- Complex numbers (which are a special case of the previous item, as discussed below).
- Clifford Algebra in higher dimensions.

2) If, however, you already have experience with complex numbers (and vectors), you can use the correspondences discussed below to jump-start your understanding of Clifford Algebra.

— **Complex Numbers** —	— **Clifford Algebra** —
A complex number has real part and an imaginary part.	A multivector has a scalar part, and a vector part, and a bivector part, et cetera. For an explanation of these concepts, related concepts, terminology, et cetera, including the geometric and pictorial representation of these objects, see reference 1, reference 2, reference 3, and reference 4.
Adding a real number to an imaginary number is like adding apples and oranges. But there's nothing wrong with that. People do it all the time.	Adding a scalar and a vector plus a bivector etc. is like adding apples and oranges plus pears et cetera. But there's nothing wrong with that. You can't safely *compare* apples and oranges, but that's a separate issue.
Due to a quirk in the terminology, the imaginary part of a complex number does not refer to an imaginary number, but refers instead to the real number multiplying i. If $z = x + iy$ (where x and y are real) the imaginary part of z is y.	The terminology for Clifford Algebra does not share this quirk. The vector part is a vector. The bivector part is a bivector.
Consider the subset of complex numbers where the imaginary part is restricted to be zero. This corresponds to the plain old scalars.	Consider the subset of multivectors where everything but the grade=0 part is restricted to be zero. This corresponds to the plain old scalars.
Multiplication is associative and distributes over addition.	Multiplication is associative and distributes over addition.
The ordinary product of two complex numbers p and q is written $p\,q$ without any special operator symbol.	The geometric product of two multivectors P and Q is written $P\,Q$ without any special operator symbol. This geometric product is primary and fundamental. Other operations, including dot product and wedge product, will be defined in terms of the geometric product.
We postulate the existence of the imaginary unit (i) whereupon all the other complex numbers can be created by multiplication and addition.	We postulate the existence of some number of vectors ($\gamma_1, \gamma_2 \ldots$) whereupon all the other multivectors can be created by multiplication and addition.

The imaginary unit (i) does *not* correspond to a Clifford-Algebra vector, but rather a bivector: $x + iy$ corresponds to $x + \gamma_1\gamma_2\, y$.

Consider the Clifford Algebra in two dimensions and restrict attention to the subset of multivectors where the grade=1 part is zero. This subset is closed under multiplication. This subset is isomorphic to the complex numbers.

Multiplication is commutative.

Multiplication of vectors is commutative if the vectors are colinear. Multiplication of vectors is anticommutative ($pq = -qp$) if the vectors are orthogonal. In general multiplication is neither commutative nor anticommutative. Most things in the real world are non-commutative. Putting on your socks doesn't commute with putting on your shoes.

The complex number system doesn't have vectors, just grade=0 real things and grade=2 imaginary things. The imaginary unit (i) is not constructed from vectors but exists by fiat.

A *blade* of grade r is defined to be the product of r mutually-orthogonal vectors. If you have a bunch of vectors but don't know for sure that they are orthogonal, you can express the wedge product in terms of permuted geometric products:

$$q_1 \wedge q_2 \wedge q_3 \cdots q_r := \frac{1}{r!} \sum_{\pi} \text{sign}(\pi) q_{\pi(1)}\, q_{\pi(2)}\, q_{\pi(3)} \cdots q_{\pi(r)} \tag{1}$$

where the sum runs over all $r!$ possible permutations π, and $\text{sign}(\pi)$ is $+1$ for even permutations and -1 for odd permutations. This will be a blade of grade r if the vectors are linearly independent; otherwise it will be zero.

So we see that the wedge product is the completely antisymmetric product. For a discussion of the physical interpretation, in terms of area of parallelograms and volume of parallelepipeds, see reference 5.

The wedge product is associative and distributes over addition.

We have not assumed the existence of a right-handed basis. Indeed we have not assumed the existence of a basis of any kind.

Some complex numbers are pure real. Some complex numbers are pure imaginary.

We say a multivector is homogeneous if it is a blade or a sum of blades all of the same grade. In $D = 3$ or less, every homogeneous multivector is a blade. In $D = 4$ and higher, you can have things like $\gamma_1\gamma_2 + \gamma_3\gamma_4$ which is homogeneous but not a blade.

We can select out the real part $\Re(z)$ or the imaginary part $\Im(z)$ for any complex number z.

We can select out the grade=r part $\langle M \rangle_r$ for any multivector M.

We know how to form the complex conjugate of a complex number: $(2 + 5i)* = (2 - 5i)$

We know how to form the reverse of a multivector: For every term that is a product of vectors, write the factors in reverse order: $(2 + 5\gamma_1\gamma_2)\sim = (2 + 5\gamma_2\gamma_1)$

Given two complex numbers p and q, their wedge product is $p \wedge q = \frac{1}{2}[(pq) - (pq)*]$. This is pure imaginary, and constitutes the high-grade piece of the ordinary product. This has norm $|p||q|\sin(\theta)$ where θ is the angle between the two vectors, which agrees with the ideas in reference 5.

Quite generally, the wedge product will be the *high-grade* part of the geometric product. That is, if A has grade=r and B has grade=s, then $A \wedge B = \langle A\,B\rangle_{r+s}$. This is a consequence of the previous definitions, since only the high-grade piece will survive the antisymmetrization. It is often much easier to pick out the high-grade piece by eye than to actually carry out the sum indicated in equation 1. (If you expand all vectors in terms of components, using an orthogonal basis, it is particularly easy to be certain of the grade of any given term.)

Given two complex numbers p and q, their dot product is $p \cdot q = \frac{1}{2}[(pq) + (pq)*]$. This is pure real, and constitutes the low-grade piece of the ordinary product. This has norm $|p||q|\cos(\theta)$ where θ is the angle between the two vectors.

Quite generally, we define the dot product as follows: The dot product of a scalar with anything is zero. Otherwise, the dot product of two multivectors is the low-grade piece of the geometric product. That is, if A has grade=r and B has grade=s, then $A \cdot B = \langle A\,B\rangle_{|r-s|}$.

The ordinary product can be written as the sum of the wedge product and dot product: $p\,q = p \wedge q + p \cdot q$

If either P or Q is a vector, then $P\,Q = P \wedge Q + P \cdot Q$. In general, though, dot and wedge don't exhaust the possibilities. If P has grade r and Q has grade s, the geometric product will contain contributions of every grade from $|r-s|$ up to $r+s$, counting by twos.

The product of a complex number with its conjugate is a real scalar.

The product of a blade with its reverse is automatically a scalar. We assume all scalars are real, because anything you could ever want to do with complex numbers can be done within the Clifford Algebra formalism.

We use this to define the squared norm of a complex number: if $z = x + iy$ then

$$|z|^2 := z\,z* = x^2 + y^2 \tag{2}$$

We use this to define the squared norm of a multivector: if $M = a + b\gamma_1 + c\gamma_2\gamma_3$ where a, b, and c are scalars, then

$$||M||^2 := \langle M\,M\sim\rangle_0 = a^2 + b^2 + c^2 \tag{3}$$

2 References

1. Stephen Gull, Anthony Lasenby, and Chris Doran, "The Geometric Algebra of Spacetime" http://www.mrao.cam.ac.uk/˜clifford/introduction/intro/intro.html

2. Richard E. Harke, "An Introduction to the Mathematics of the Space-Time Algebra" http://www.harke.org/ps/intro.ps.gz

3. David Hestenes, "Oersted Medal Lecture 2002: Reforming the Mathematical Language of Physics"
Abstract: http://modelingnts.la.asu.edu/html/overview.html
Full paper: http://modelingnts.la.asu.edu/pdf/OerstedMedalLecture.pdf

4. John Denker "Introduction to Clifford Algebra" ./clifford-intro.htm

5. John Denker, "Area and Volume of Parallelograms and Parallelepipeds" ./area-volume.htm