1 Introduction

The purpose of this note is to derive Euler's equation for fluid flow (equation 18) without cheating, just using sound physics principles such as conservation of mass, conservation of momentum, and Newton's laws. (There are way too many unsound derivations out there.)

To set the stage, consider the example shown in figure 1. The top half is a snapshot of the fluid at some initial time t_0 and the bottom half is a snapshot at some slightly later time t_1. One parcel of fluid has been marked with blue dye, and another parcel has been marked with red dye. The rectangle represents an imaginary box, also called the "control volume" ... we will be particularly interested in what is happening within this box.

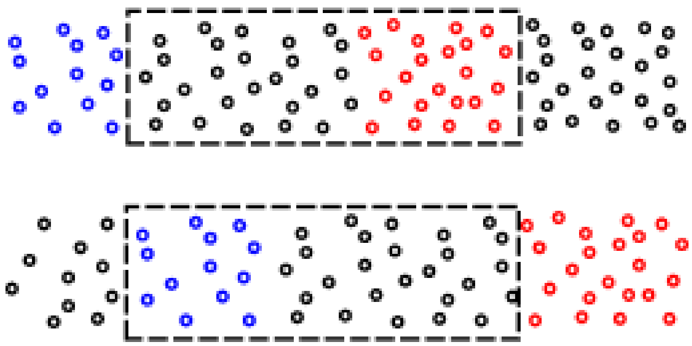

Figure 1: Flow

There are some qualitative observations that we can make immediately:

- The fluid is flowing left-to-right.
- The density depends on position: The red parcel is denser than the blue parcel.
- The density depends on time: The density in the control volume decreases as time passes from t_0 to t_1.
- The velocity depends on position: during the time interval in question, the right edge of the blue parcel moves a relatively short distance, while the right edge of the red parcel moves a relatively long distance.
- The fluid is conserved. The total number of red particles does not change, and the total number of blue particles does not change. (Some particles that flow out of the frame of the diagram are not accounted for, but all particles – red, blue, or black – that affect the control volume are accounted for.)

2 Conservation of Mass

Suppose we have a fluid with local density $\rho(t, x, y, x)$ and local velocity $\mathbf{v}(t, x, y, z)$. Consider a control volume V (not necessarily small, not necessarily rectangular) which has boundary S. The total mass in this volume is

$$M = \int \rho \mathrm{d}V \tag{1}$$

The rate-of-change of this mass is just

$$\frac{\partial M}{\partial t} = \int \frac{\partial \rho}{\partial t} \mathrm{d}V \tag{2}$$

The only way such change can occur is by stuff flowing across the boundary, so

$$\frac{\partial M}{\partial t} = \int \rho\mathbf{v}\cdot\mathbf{d}S \tag{3}$$

We can change the surface integral into a volume integral using Green's theorem, to obtain

$$\frac{\partial M}{\partial t} = -\int \nabla\cdot(\rho\mathbf{v})\mathrm{d}V \tag{4}$$

Compare equation 2 with equation 4. They are equal no matter what volume V we choose, so the integrands must be pointwise equal. This gives us an expression for the local conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{5}$$

We can understand this expression by referring back to figure 1. At the right edge of the figure, the fluid is relatively dense and has a relatively high velocity, causing a large outflow of fluid from the control volume (the red parcel). At the left edge, the density is relatively low and the velocity is relatively low, causing a small inflow of fluid. The x-derivative of ρv_x is positive. This is the crucial contribution to $\nabla\cdot(\rho\mathbf{v})$; the other two contributions vanish in this example. We have a net outflow of fluid, which causes a decrease in the density of the fluid in the control volume, in accordance with equation 5.

Equation 5 is sometimes called a *continuity equation*. It is one way to express conservation of mass.

3 Conservation of Momentum

We can go through the same process for momentum instead of mass. We use Π to represent momentum, to avoid conflict with P which represents pressure. The total momentum in the control volume is:

$$\Pi_i = \int \rho v_i \mathrm{d}V \tag{6}$$

where the index i runs over the three components of the momentum. The rate-of-change thereof is just

$$\frac{\partial \Pi_i}{\partial t} = \int \frac{\partial(\rho v_i)}{\partial t}\mathrm{d}V \tag{7}$$

We (temporarily) assume there are no applied forces (i.e. no gravity etc.) and no pressure (e.g. a fluid of non-interacting dust particles). We also assume viscous forces are negligible. Then, the only way a momentum-change can occur is by momentum flowing across the boundary:

$$\frac{\partial \Pi_i}{\partial t} = \int (\rho v_i)\mathbf{v}\cdot\mathbf{d}S = \int (\rho v_i) v_j \mathrm{d}_j S \tag{8}$$

We are expressing dot products using the Einstein summation convention, i.e. implied summation over repeated dummy indices, such as j in the previous expression.

We can change the surface integral into a volume integral using Green's theorem, to obtain

$$\frac{\partial \Pi_i}{\partial t} = -\int \nabla_j(\rho v_i v_j)\mathrm{d}V \tag{9}$$

Compare equation 7 with equation 9. They are equal no matter what volume V we choose, so the integrands must be pointwise equal. This gives us an expression for the local conservation of momentum:

$$\frac{\partial \Pi_i}{\partial t} = \frac{\partial(\rho v_i)}{\partial t} = -\nabla_j(\rho v_i v_j) \tag{10}$$

We can understand this equation as follows: each component of the momentum-density ρv_i (for each i separately) obeys a local conservation law. There are strong parallels between equation 5 and equation 10.

Note that the ∇_j operator on the RHS is differentiating *two* velocities (v_i and v_j) only one of which undergoes dot-product summation (namely summation over j). Using vector-component notation (such as $\nabla_j v_j$) is a bit less elegant than using pure vector notation (such as $\nabla \cdot \mathbf{v}$) but in this case it makes things clearer.

We now consider the effect of pressure. It contributes a force on the particles in the control volume, namely

$$\begin{array}{lll} F_i & += & \int P \mathrm{d}_i S \\ & = & -\int \nabla_i P \mathrm{d}V \end{array} \tag{11}$$

A uniform gravitational field contributes another force, namely

$$F_i += \int \rho g_i \mathrm{d}V \tag{12}$$

These forces contribute to changing the momentum, by Newton's second law:

$$\frac{\mathrm{d}\Pi'_i}{\mathrm{d}t} = F_i \tag{13}$$

Note the tricky notation: we write $\mathrm{d}/\mathrm{d}t$ rather than $\partial/\partial t$, and Π' rather than Π, to remind ourselves that Newton's laws apply to particles, *not* to the control volume itself. The rate-of-change of Π, the momentum *in the control volume*, contains the Newtonian contributions (equation 11 and equation 12 via equation 13) plus the flow contributions (equation 10).

Combining all the contributions, we obtain the main result, the equation of motion:

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla_j(\rho v_i v_j) = -\nabla_i P + \rho g_i \tag{14}$$

This way of expressing the equation of motion has many advantages. It has a somewhat elegant symmetrical form. It expresses conservation of momentum in a way that is strongly analogous to conservation of mass (equation 5). This $D = 3$ expression can readily be generalized to give an expression that is valid in $D = 1{+}3$ spacetime. It assumes viscous forces are negligible, but it is otherwise rather general.

However, one sometimes encounters other ways of expressing the same equation of motion. If we expand the LHS we get

$$\frac{\rho\partial(v_i)}{\partial t} + \frac{v_i\partial(\rho)}{\partial t} + v_i\nabla_j(\rho v_j) + \rho v_j\nabla_j(v_i) = -\nabla_i P + \rho g_i \tag{15}$$

where the second and third terms cancel because of conservation of mass (equation 5), leaving us with

$$\rho\frac{\partial(v_i)}{\partial t} + \rho v_j\nabla_j(v_i) = -\nabla_i P + \rho g_i \tag{16}$$

Then, if we convert from component notation to vector notation, we get a version of Euler's equation,

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P + \rho\mathbf{g} \tag{17}$$

If we divide through by ρ, we obtain a slightly more traditional version of Euler's equation:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{-\nabla P}{\rho} + \mathbf{g} \tag{18}$$

This is a famous and important equation. It assumes viscous forces are negligible, but it is otherwise rather general.

It is sometimes convenient to introduce the notion of *head*, defined as:

$$\begin{aligned} H &:= P - \int \rho \mathbf{g} \cdot \mathrm{d}z \\ &\approx P + \rho|\mathbf{g}|z \end{aligned} \tag{19}$$

The approximation in the second line of equation 19 is valid when the density ρ is approximately constant over the region of interest. (Throughout this document we assume $\mathbf{g}$ is constant.) The dot product on the first line contributes a minus sign, because $\mathbf{g}$ is directed downwards while $\mathrm{d}z$ is directed upwards.

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla H \tag{20}$$

Roughly speaking, we see that the head plays the role of some sort of potential energy. The pressure and the gravitational potential energy contribute equally to the head. The head gives rise to a force on the RHS of equation 20, roughly in accordance with the principle of virtual work.

4 Remarks

A few words of warning about equation 17: the first term on the LHS looks reminiscent of Newton's law: a mass-density times an acceleration. And the RHS looks like a force-density. But that is *not* a good way to think about it, for reasons that will become clear in a moment. A lot of people who ought to know better (e.g. Landau and Lifschitz) purport to derive Euler's equation by analogy to Newton's second law. That is, they start with $\rho \mathrm{d}\mathbf{v}/\mathrm{d}t$, set it equal to the force-density and then correct it with the flow term (the second term on the LHS of equation 17). Alas, that's logically unsound, and I don't see any way to fix it. The alternative is to use the correct expression for the force density, i.e. equation 14, which has ρ *inside* the derivative $\partial(\rho\mathbf{v})/\partial t$.

Equation 14 is worth emphasizing. It is useful unto itself, it is easy to remember in analogy to equation 5, and it is a valid starting point for a derivation of the traditional Euler equation. It also has the nice property that, with minor modifications, it can be put in relativistically-correct form.

5 Bernoulli's Formula

To derive Bernoulli's formula, we assume steady flow and approximately constant density, and then integrate equation 20 along a streamline.

Because the flow is steady, the first term on the LHS of equation 20 vanishes.

Because we are integrating along a streamline, we can replace $(\mathbf{v} \cdot \nabla)\mathbf{v}$ with $\frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v})$. The two expressions are not numerically equal, but the difference is a vector pointing crosswise to the streamline, and contributes nothing to the integral. To see this, note that $(\mathbf{v} \cdot \nabla)\mathbf{v}$ contains contributions from changes in the magnitude of $\mathbf{v}$ and in the direction of $\mathbf{v}$, whereas $\nabla(\mathbf{v} \cdot \mathbf{v})$ only cares about the magnitude of $\mathbf{v}$. The trick is that a change in direction of $\mathbf{v}$ is represented by a vector transverse to $\mathbf{v}$, hence transverse to the streamline, so it contributes nothing to the integral.

We integrate from point B ("before") to point A ("after") along a streamline:

$$\int_B^A (\frac{1}{2}\rho\nabla(\mathbf{v} \cdot \mathbf{v}) + \nabla H) \cdot \mathrm{d}s = 0 \tag{21}$$

We want to move ρ outside the integral, so we assume that point B and point A are not too far apart, and that ρ is slowly varying. You can quantify this by expanding ρ in a power series as a function of pressure, and keeping only the zeroth order term. (The higher-order terms allow you to estimate the accuracy of the approximation.)

$$\begin{aligned} \tfrac{1}{2}\rho v^2 + H &= \text{const} \\ \tfrac{1}{2}\rho v^2 + P &\approx \text{const} \end{aligned} \tag{22}$$

It must be emphasized that equation 22 is only valid for point A near point B along a particular streamline. In particular, the constant on the RHS may be (and often is) different for different streamlines.

> Sometimes, in some special case, you may be able to ascertain that the constant in equation 22 is the same for some group of streamlines, in which case you can conclude that $\frac{1}{2}\rho v^2 + H$ has the same value for all streamlines in that group. Just remember that this is not the general case.

In the last line of equation 22 we approximate H by P, which is valid when the region of interest is sufficiently limited in the z direction that gravitational terms do not contribute significantly to the head. This is commonly the case when considering the flow of air over a wing, or the flow of air through a carburetor.

Equation 22 is the "baby Bernoulli" formula. More sophisticated versions of this formula can be derived, for instance versions that relax the assumption about constant density. See reference 2 and reference 3.

6 References

1. John Denker, "Conservative Flow in Spacetime" ./conservative-flow.htm
2. John Denker, "Airfoils and Airflow" [Chapter 3 of **See How It Flies** ../how/htm/airfoils.html
3. Richard von Mises, **Theory of Flight**, (1945; Dover reprint 1959) ISBN 0 486 60541 8.