

# 1 Magnetic Field of a Long Straight Wire

The task for today is to calculate the electromagnetic field associated with the flow of current in a long straight wire. We choose to do this the modern way, representing the electromagnetic field as a bivector (not as a vector or pseudovector). We will make direct use of the modern form of the Maxwell equations (equation 2).

By way of contrast: We are not going to write the Maxwell equations in the old-fashioned way, in terms of cross products etc., and we are not going to represent the magnetic field as a vector (or pseudovector).

So, let's begin. We make the following assumptions:

- There is a wire running along the  $z$  axis.
- The wire has constant radius  $R$ .
- The wire carries total current  $I$ .
- The current density is uniform across the cross section of the wire.
- This wire is the only significant source of magnetic fields, and in particular the field goes to zero far from the wire.
- This is a magnetostatics problem, so nothing changes as a function of time.

Therefore the current density is

$$\begin{aligned} J &= \frac{I}{\pi R^2} && \text{inside the wire} \\ J &= 0 && \text{outside} \end{aligned} \tag{1}$$

Now we can plug this into the Maxwell equation(s) and turn the crank. We choose to use Clifford Algebra ideas; an overview can be found in reference 1 and references therein. That allows us to write everything we need to know about the electromagnetic field  $F$  using one simple equation:

$$\nabla F = \frac{1}{c\epsilon_0} J \tag{2}$$

That equation can be shown to be entirely equivalent to the old-fashioned form of the Maxwell equations, as demonstrated in reference 2.

We find it convenient to choose a set of basis vectors. We choose  $\gamma_1$  as the basis vector in the  $x$  direction,  $\gamma_2$  as the basis vector in the  $y$  direction, and  $\gamma_3$  as the basis vector in the  $z$  direction. These basis vectors are orthonormal. Nowhere do we assume that they form a right-handed set.

Using this basis, we can expand the  $\nabla$  operator in terms of its components, namely

$$\nabla = \gamma_1 (\partial/\partial x) + \gamma_2 (\partial/\partial y) \tag{3}$$

where we have omitted the  $t$  and  $z$  derivatives because they vanish by symmetry.

We use the symbol  $I$  to denote the *vector* current. If you want the corresponding scalar, that is denoted  $I_m$ , where the subscript  $m$  means "in the direction of the meter." For simplicity, we hereby assume that the meter is oriented so that it measures current in the  $+z$  (not  $-z$ ) direction, which means that

$$\begin{aligned} I_m &= I \cdot \gamma_3 \\ I &= I_m \gamma_3 \end{aligned} \tag{4}$$

Let's start by solving the equation inside the wire. We are looking for some  $F$  that has a constant derivative. We shouldn't be too surprised if it is linear in the  $x$  and  $y$  coordinates, since the  $x$ -derivative of  $x$  is a constant and the  $y$ -derivative of  $y$  is a constant.

A good guess for  $F$  would be to write

$$F = ax\gamma_1\gamma_3 + by\gamma_2\gamma_3 + G \tag{5}$$

Where  $a$  and  $b$  are constants to be determined,  $G$  is any solution to the homogenous equation  $\nabla F = 0$ . We will have more to say about  $G$  anon.

Plugging this into equation 2 we can find suitable values for  $a$  and  $b$ , with the result that the field inside the wire is

$$F = \frac{1}{2\pi\epsilon_0 c} x \frac{I_m}{R^2} \gamma_1 \gamma_3 + \frac{1}{2\pi\epsilon_0 c} y \frac{I_m}{R^2} \gamma_2 \gamma_3 + G \quad [\text{inside}] \quad (6)$$

Next, let's consider what happens outside the wire. We shouldn't be too surprised if the field falls off in inverse proportion to the distance from the axis. This is what we would expect based on a Gauss's law argument. Those who are not familiar with Gauss's law can verify that the following guess:

$$F = \frac{1}{2\pi\epsilon_0 c} x \frac{I_m}{x^2 + y^2} \gamma_1 \gamma_3 + \frac{1}{2\pi\epsilon_0 c} y \frac{I_m}{x^2 + y^2} \gamma_2 \gamma_3 + G \quad [\text{outside}] \quad (7)$$

is in fact a very good guess. It satisfies the Maxwell equation outside the wire ( $\nabla F = 0$ ) and also agrees with the other solution (equation 6) where they meet at the boundary of the wire ( $x^2 + y^2 = R^2$ ).

At this point we will observe that  $G = 0$  produces a satisfactory solution for  $F$ . In fact, although we won't prove it here, the solution is unique. Given the way we have chosen  $a$  and  $b$ ,  $G = 0$  is the only  $G$  function that causes equation 7 to satisfy the boundary conditions (vanishing at large distances) and the required symmetries (independent of  $z$ , and rotationally invariant in the plane perpendicular to  $z$ ) and is time-independent.

You should not, however, get into the habit of assuming that  $G = 0$  in problems of this sort. In this case  $G$  is zero only because we chose lucky values for  $a$  and  $b$  in equation 5. If we had instead chosen  $a' = 2a$  and  $b' = 0$ , then a nonzero  $G$  would have been necessary (and sufficient) to give us a satisfactory solution for  $F$ .

We can tidy up equation 7 quite a bit by writing it in a more vector-friendly basis-independent form, as follows:

We start by defining the position vector in all generality:

$$r := x\gamma_1 + y\gamma_2 + z\gamma_3 \quad (8)$$

The component of  $r$  parallel to  $I$  is given by the usual Gram-Schmidt formula:

$$r_I := I \frac{r \cdot I}{I \cdot I} \quad (9)$$

and you can verify that  $I \cdot r_I = I \cdot r$ .

The component of  $r$  perpendicular to  $I$  is what's left, namely, in all generality:

$$r_{\perp} := r - r_I \quad (10)$$

Applying these general definitions to our specific geometry, namely cylindrical coordinates with the axis in the  $\gamma_3$  direction, we find:

$$\begin{aligned} r_I &= z\gamma_3 \\ r_{\perp} &= x\gamma_1 + y\gamma_2 \end{aligned} \quad (11)$$

This  $r_{\perp}$  is the radius vector in cylindrical coordinates, always perpendicular to the  $z$ -axis. The square of its length is  $r_{\perp}^2 = x^2 + y^2$ .

We also recall the definition of the  $I$  vector, equation 4. Putting the pieces together, we see that the field outside a long straight wire is

$$\begin{aligned} F &= \frac{1}{2\pi\epsilon_0 c} \frac{r_{\perp}}{r_{\perp}^2} I \\ &= \frac{1}{2\pi\epsilon_0 c} \frac{1}{r_{\perp}} I \end{aligned} \quad (12)$$

where we have used the fact that  $r_{\perp}/r_{\perp}^2$  is the reciprocal of  $r_{\perp}$ , as you can verify by multiplying them together.

It might be even better to write this as

$$F = \frac{1}{2\pi\epsilon_0 c} \frac{1}{r_{\perp}} \wedge I \quad (13)$$

which is equivalent since by construction  $r_{\perp}$  is perpendicular to  $I$ .

The orientation of this field is shown by the blue rectangular bivectors in figure 1.

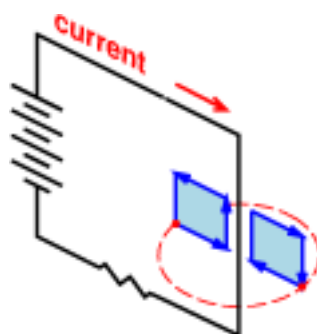


Figure 1: Field of a Long Straight Wire

We can combine the expression for the field inside the wire and the field outside the wire as follows:

$$F = \frac{1}{2\pi\epsilon_0 c} \frac{r_{\perp} \wedge I}{\max(R^2, r_{\perp}^2)} \quad (14)$$

## 2 Remarks

Remember that multiplication of perpendicular vectors is anticommutative, so if you write the factors in the other order it picks up a minus sign; for example,  $I \wedge (1/r_{\perp}) = -I \wedge (1/r_{\perp})$ . This means that the sense of circulation of  $F$  is such that the edge nearest the wire is directed opposite to the flow of the current.

It is very useful to notice that  $R$  (the radius of the wire) does not appear in the expressions for the field outside the wire. The only thing that matters is the total current.

## 3 Stokes's Theorem and Ampère's Law

First, suppose we draw a circle in the  $XY$  plane, centered on the wire, as (for example) the red, dashed circle in figure 1. The circumference of this circle is  $2\pi r_{\perp}$ .

Next, note that we can rewrite equation 13 in the suggestive form:

$$F = \frac{1}{\epsilon_0 c} \frac{1}{2\pi r_{\perp}} \wedge I \quad (15)$$

and it is absolutely not a coincidence that  $2\pi r_{\perp}$  occurs in both the expression for the circumference and the expression for the field.

In fact there is a very general mathematical theorem that goes as follows: For almost any  $F$  you can think of – including but not limited to the electromagnetic field  $F - \nabla \wedge F$  can be considered some sort of flow-density.

Then, given any patch of surface  $S$ , the total flux of  $\nabla \wedge F$  flowing through the surface is equal to the line integral of  $F$  itself, integrated along the boundary of the surface. That is:

$$\int_S \nabla \wedge F = \int_{\partial S} F \quad (16)$$

This is called Stokes's Theorem. For the next level of detail, see reference 7.

As a corollary of this theorem, we can use equation 2 which gives us a nice expression for  $\nabla F$  in terms of the current density. Plugging in, we obtain:

$$\int_{\partial S} F = \frac{1}{\epsilon_0 c} I_S \quad (17)$$

This is called Ampère's Law. It says that the integral of the magnetic field (integrated around some loop  $\partial S$ ) is equal to the current flowing through the area  $S$  bounded by the loop, with a factor of  $(1/\epsilon_0 c)$  thrown in to make the units come out right.

We have made use of the fact that this is a magnetostatics problem, so that  $\nabla F$  equals  $\nabla \wedge F$ .

The work we did in section 1 shows that this result is true for the special case of a circular loop around a long straight wire. Ampère's Law is upheld everywhere outside and inside the wire, in accordance with equation 14.

With some additional geometrical arguments you can convince yourself that equation 16 must be true in general, for any shape of loop and for any distribution of current density; see reference 8.

In some sense, calculating the magnetic field by direct use of equation 2 is the hard way to do it ... but we needed to do it that way once, in order to explain the origin of Ampère's Law. However, now that we have Ampère's Law, we can use it to our advantage. It makes equation 15 well-nigh unforgettable. The field is proportional to the current and inversely proportional to  $2\pi r_\perp$  in accordance with the circumference of a circle via Stokes's Theorem, and there is a factor of  $(1/\epsilon_0 c)$  inherited from the RHS of equation 2.

## 4 Field Inside a Long Solenoid

Another nice use of Ampère's Law is to calculate the field inside a long solenoid. The situation is depicted in figure 2.

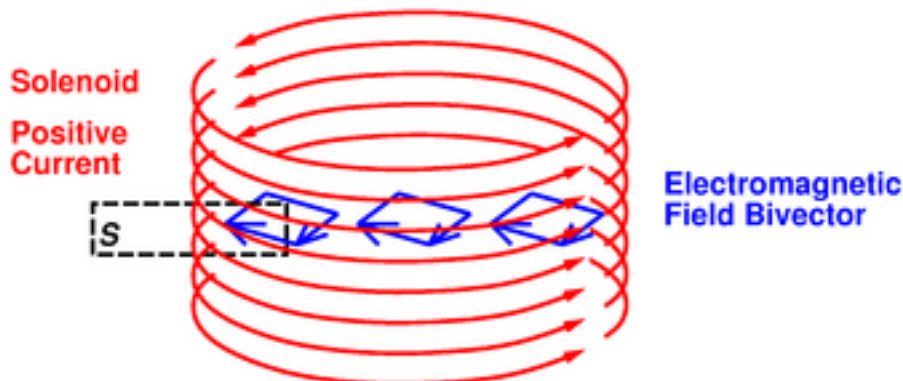


Figure 2: Field Inside a Long Solenoid

The electromagnetic field bivector  $F$  is purely magnetic, i.e. it is aligned in a purely spatial direction in spacetime, in the frame comoving with the solenoid. The bivector lies in the plane of symmetry, i.e. perpendicular to the axis of the solenoid. The field is uniform everywhere inside the solenoid, provided we don't get too near the ends.

The orientation is such that if the current in the solenoid carries positive current circulating in one direction, the field bivector  $F$  circulates in the opposite direction.

The magnitude of the field  $F$  is:

$$\begin{aligned} F &= \frac{1}{\epsilon_0 c} \frac{I}{L} \\ &= \frac{1}{\epsilon_0 c} \frac{N I_w}{L} \end{aligned} \quad (18)$$

where  $(I/L)$  is the current per unit length. The second line of the equation applies to the very common case where the circulating current is created by a wire carrying current  $I_w$  and there are  $N/L$  turns per unit length; each turn makes its own contribution to the total circulating current.

This result can be obtained in the familiar way, by applying Ampère's Law to a surface  $S$  such as the one bounded by the black dashed line in figure 2.

All these results are based on equation 2. At no point have we found it necessary or even useful to calculate the old-fashioned magnetic field pseudovector. On the other hand, in the spirit of the correspondence principle, if you wish to compare equation 18 with the old-fashioned bivector expression, keep in mind that the magnetic component of  $F$  is, roughly speaking  $c\mathbf{B}$ , not  $\mathbf{B}$ .

## 5 References

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